

Representations of A-type Hecke algebras¹

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Abstract. We review some facts about the representation theory of the Hecke algebra. We adapt for the Hecke algebra case the approach of [1] which was developed for the representation theory of symmetric groups. We justify an explicit construction of the idempotents in the Hecke algebra in terms of Jucys-Murphy elements. Ocneanu's traces for these idempotents (which can be interpreted as q-dimensions of corresponding irreducible representations of quantum linear groups) are presented.

1 Introduction

Main statements of the representation theory of Hecke algebras are known mostly due to the works by V.Jones, I.V.Cherednik, G.Murphy, R.Dipper and G.James, H.Wenzl, a.o. (see, e.g., [2] – [5]). In this report the approach of [1], developed for the representation theory of symmetric groups, is generalized to the case of the *A*-type Hecke algebras. Certain propositions below are given without proofs due to lack of space and, also, because the corresponding statements for Hecke algebras are proved like those for symmetric groups.

The importance of the theory of the *A*-type Hecke algebra H_M is that H_M is the centralizer of the action of general linear quantum groups $U_q(gl(N))$ in the tensor powers $V^{\otimes M}$ of the vector representation V of $U_q(gl(N))$. We have shown recently [6] that an

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arbitrary representation of the Hecke algebra H_M defines an integrable model on a chain with M sites. This fact demonstrates the importance of the representation theory of the Hecke algebra in the theory of integrable models also.

2 A-Type Hecke algebras and Jucys - Murphy elements

A braid group \mathcal{B}_{M+1} is generated by Artin elements σ_i ($i = 1, \dots, M$) subject to relations:

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \quad \sigma_i \sigma_j = \sigma_j \sigma_i \text{ for } |i - j| > 1. \quad (2.1)$$

An A -Type Hecke algebra $H_{M+1}(q)$ (see e.g. [2] and Refs. therein) is a quotient of the group algebra of the braid group \mathcal{B}_{M+1} by an additional relation

$$\sigma_i^2 - 1 = (q - q^{-1}) \sigma_i, \quad (i = 1, \dots, M). \quad (2.2)$$

Here $q \in \mathbf{C} \setminus \{0\}$ is a parameter. The group algebra of \mathcal{B}_{M+1} (2.1) has an infinite dimension while its quotient H_{M+1} is finite dimensional. It can be shown (see e.g. [5]) that H_{M+1} is spanned linearly by $(M + 1)!$ elements, e.g., those which appear in the expansion of the special operator

$$\Sigma_{1 \rightarrow M+1} = f_{1 \rightarrow M+1} f_{1 \rightarrow M} \cdots f_{1 \rightarrow 2} f_{1 \rightarrow 1},$$

where $f_{1 \rightarrow n}$ are 1-shuffles defined inductively by $f_{1 \rightarrow 1} = 1$, $f_{1 \rightarrow n+1} = 1 + f_{1 \rightarrow n} \sigma_n$. Below we assume that $q \neq \exp(2\pi i n/m)$, $n, m \in \mathbf{Z}$ (q is "generic"); for these values of q , there exists an isomorphism between the algebra $H_{M+1}(q)$ and the group algebra of the symmetric group S_{M+1} (the case $q = \pm 1$ is exceptional, in this case H_{M+1} = group algebra of S_{M+1}).

An essential information about a finite dimensional semisimple algebra \mathcal{A} is contained in the structure of its regular bimodule which decomposes into direct sums: $\mathcal{A} = \bigoplus_{\alpha=1}^s \mathcal{A} \cdot e_\alpha$, $\mathcal{A} = \bigoplus_{\alpha=1}^s e_\alpha \cdot \mathcal{A}$ of left and right submodules (ideals), respectively (left- and right-Peirce decompositions). Here the elements $e_\alpha \in \mathcal{A}$ ($\alpha = 1, \dots, s$) are mutually orthogonal idempotents: $e_\alpha e_\beta = \delta_{\alpha\beta} e_\alpha$, resolving the identity operator: $1 = \sum_{\alpha=1}^s e_\alpha$. There are two important decompositions of the identity operator and correspondingly two sets of the idempotents in \mathcal{A} :

(1) *Primitive idempotents*. An idempotent e_α is *primitive* if it can not be further resolved into a sum of nontrivial mutually orthogonal idempotents.

(2) *Primitive central idempotents*. An idempotent e'_β is *primitive central* if it is primitive in the class of central idempotents.

For the A -type Hecke algebra $H_{M+1}(q)$ a set of elements $\{y_i\}$ ($i = 1, \dots, M + 1$) is defined inductively: $y_1 = 1$, $y_{i+1} = \sigma_i y_i \sigma_i$. These elements are called *Jucys - Murphy elements* and can be written (using the Hecke condition (2.2) and the braid relation (2.1)) in the form

$$y_i = \sigma_{i-1} \dots \sigma_2 \sigma_1^2 \sigma_2 \dots \sigma_{i-1} = (q - q^{-1}) \sum_{k=1}^{i-1} \sigma_k \dots \sigma_{i-2} \sigma_{i-1} \sigma_{i-2} \dots \sigma_k + 1. \quad (2.3)$$

Sometimes it is more convenient to use elements $(y_i - 1)/(q - q^{-1})$ which, due to (2.3), have a nontrivial classical limit ($q \rightarrow 1$). The elements y_i pairwise commute. The following statement explains the importance of the set $\{y_i\}$.

Proposition 1. *The set of Jucys - Murphy elements $\{y_i\}$ ($i = 1, \dots, M + 1$) generates a maximal commutative subalgebra Y_{M+1} in H_{M+1} .*

We construct primitive orthogonal idempotents $e_\alpha \in H_{M+1}$ as functions of the elements $y_i \in Y_{M+1}$; they are common eigenidempotents of y_i : $y_i e_\alpha = e_\alpha y_i = a_i^{(\alpha)} e_\alpha$ ($i = 1, \dots, M + 1$). We denote (as in [1], for symmetric groups) by $\text{Spec}(y_1, \dots, y_{M+1})$ the set $\{\Lambda(e_\alpha)\}$ ($\forall \alpha$) of strings of eigenvalues: $\Lambda(e_\alpha) = (a_1^{(\alpha)}, \dots, a_{M+1}^{(\alpha)})$. In view of the following inclusions of the subalgebras Y_i and $H_i(q)$:

$$\begin{gathered} H_i(q) \subset H_{i+1}(q) \\ \bigcup Y_i \subset \bigcup Y_{i+1} \end{gathered}$$

one can describe the idempotents $\in H_{i+1}$ by considering the branching of the idempotents of H_i in H_{i+1} . It can be shown that the multiplicity of this branching is equal to one and y_i are semi-simple for generic q .

We need important intertwining operators [8] (presented in another form in [3])

$$U_{n+1} = \sigma_n y_n - y_n \sigma_n \quad (1 \leq n \leq M). \quad (2.4)$$

Elements U_i satisfy relations³ $U_n U_{n+1} U_n = U_{n+1} U_n U_{n+1}$ and

$$U_{n+1} y_n = y_{n+1} U_{n+1}, \quad U_{n+1} y_{n+1} = y_n U_{n+1}, \quad [U_{n+1}, y_k] = 0 \quad (k \neq n, n+1), \quad (2.5)$$

$$U_{n+1}^2 = (q y_n - q^{-1} y_{n+1})(q y_{n+1} - q^{-1} y_n). \quad (2.6)$$

The operators U_{n+1} "permute" elements y_n and y_{n+1} (see (2.5)) which supports a statement that the center Z_{M+1} of the Hecke algebra H_{M+1} is generated by symmetric functions in $\{y_i\}$ ($i = 2, \dots, M + 1$) (to prove this fact it is enough to check relations: $[\sigma_k, y_n + y_{n+1}] = 0 = [\sigma_k, y_n y_{n+1}]$ for all $k < n + 1$).

Proposition 2. *One has*

$$\text{Spec}(y_j) \subset \{q^{2\mathbf{Z}_j}\} \quad \forall j = 1, 2, \dots, M + 1, \quad (2.7)$$

where \mathbf{Z}_j denotes the set of integers $\{1 - j, \dots, -2, -1, 0, 1, 2, \dots, j - 1\}$.

Proof. We prove (2.7) by induction. Obviously, $\text{Spec}(y_1)$ satisfies (2.7). Assume that the spectrum of y_{j-1} satisfies (2.7) for some $j \geq 2$. Consider a characteristic equation for y_{j-1} ($j \geq 2$):

$$f(y_{j-1}) := \prod_{\alpha} (y_{j-1} - a_{j-1}^{(\alpha)}) = 0 \quad (a_{j-1}^{(\alpha)} \in \text{Spec}(y_{j-1})).$$

Using properties (2.5)-(2.6) of operators U_j , we deduce

$$0 = U_j f(y_{j-1}) U_j = f(y_j) U_{2j} = f(y_j) (q^2 y_{j-1} - y_j)(y_j - q^{-2} y_{j-1}). \quad (2.8)$$

which means that $\text{Spec}(y_j) \subset (\text{Spec}(y_{j-1}) \cup q^{\pm 2} \cdot \text{Spec}(y_{j-1}))$. •

³The definition (2.4) of intertwining elements is not unique. One can multiply U_{n+1} by a function $f(y_n, y_{n+1})$: $U_{n+1} \rightarrow U_{n+1} f(y_n, y_{n+1})$. Then eqs. (2.5)-(2.6) are valid if $f(y_n, y_{n+1})f(y_{n+1}, y_n) = 1$.

3 Generalization of the approach of [1] to the Hecke algebra case

Consider a subalgebra $\hat{H}_2^{(i)}$ in H_{M+1} with generators y_i , y_{i+1} and σ_i (for fixed $i \leq M$). We investigate representations of $\hat{H}_2^{(i)}$ with diagonalizable y_i and y_{i+1} . Let e be a common eigenidempotent of y_i , y_{i+1} : $y_i e = a_i e$, $y_{i+1} e = a_{i+1} e$. Then the left action of $\hat{H}_2^{(i)}$ closes on elements $v_1 = e$ and $v_2 = \sigma_i e$ and is given by matrices:

$$\sigma_i = \begin{pmatrix} 0 & 1 \\ 1 & q - q^{-1} \end{pmatrix}, \quad y_i = \begin{pmatrix} a_i & -(q - q^{-1})a_{i+1} \\ 0 & a_{i+1} \end{pmatrix}, \quad y_{i+1} = \begin{pmatrix} a_{i+1} & (q - q^{-1})a_{i+1} \\ 0 & a_i \end{pmatrix}; \quad (3.1)$$

$a_i \neq a_{i+1}$ otherwise y_i , y_{i+1} are not diagonalizable. The matrices y_i , y_{i+1} (3.1) can be simultaneously diagonalized by a similarity transformation $y \rightarrow V^{-1}yV$, where

$$V = \begin{pmatrix} 1 & \frac{(q - q^{-1})a_{i+1}}{a_i - a_{i+1}} \\ 0 & 1 \end{pmatrix}, \quad V^{-1} = \begin{pmatrix} 1 & -\frac{(q - q^{-1})a_{i+1}}{a_i - a_{i+1}} \\ 0 & 1 \end{pmatrix}.$$

As a result we obtain

$$\sigma_i = \begin{pmatrix} -\frac{(q - q^{-1})a_{i+1}}{a_i - a_{i+1}} & 1 - \frac{(q - q^{-1})2a_ia_{i+1}}{(a_i - a_{i+1})^2} \\ 1 & \frac{(q - q^{-1})a_i}{a_i - a_{i+1}} \end{pmatrix}, \quad y_i = \begin{pmatrix} a_i & 0 \\ 0 & a_{i+1} \end{pmatrix}, \quad y_{i+1} = \begin{pmatrix} a_{i+1} & 0 \\ 0 & a_i \end{pmatrix}. \quad (3.2)$$

When $a_{i+1} = q^{\pm 2}a_i$, the 2-dimensional representation (3.2) reduces to a 1-dimensional representation with $\sigma_i \cdot e = \pm q^{\pm 1}e$, respectively. We summarize the above results as (cf. Proposition 4.1 [1]):

Proposition 3. *Let $\Lambda = (a_1, \dots, a_i, a_{i+1}, \dots, a_{M+1}) \in \text{Spec}(y_1, \dots, y_{M+1})$ be a possible spectrum of the set (y_1, \dots, y_{M+1}) which corresponds to a primitive idempotent $e_\Lambda \in H_{M+1}$. Then $a_i = q^{2m_i}$, where $m_i \in \mathbf{Z}_i$ (see Prop. 2) and (a) $a_i \neq a_{i+1}$ for $i \leq M$; (b) if $a_{i+1} = q^{\pm 2}a_i$ then $\sigma_i \cdot e_\Lambda = \pm q^{\pm 1}e_\Lambda$; (c) if $a_{i+1} \neq q^{\pm 2}a_i$ then*

$$\Lambda' = (a_1, \dots, a_{i+1}, a_i, \dots, a_{M+1}) \in \text{Spec}(y_1, \dots, y_{M+1}) \quad (3.3)$$

and the left action of the elements σ_i, y_i, y_{i+1} in the linear span of $v_\Lambda = e_\Lambda$ and $v_{\Lambda'} = \sigma_i e_\Lambda + \frac{(q - q^{-1})a_{i+1}}{a_i - a_{i+1}} e_\Lambda$ is given by (3.2).

Proposition 4. *Consider the string $\Lambda = (a_1, \dots, a_n)$ of numbers $a_i = q^{2m_i}$, where $m_i \in \mathbf{Z}_i$ (see Prop. 2). Then $\Lambda = (a_1, a_2, \dots, a_n) \in \text{Spec}(y_1, y_2, \dots, y_n)$ iff Λ satisfies the following conditions ($z \in \mathbf{Z}$)*

- (1) $a_1 = 1$;
- (2) $a_j = q^{2z} \Rightarrow \{q^{2(z+1)}, q^{2(z-1)}\} \cap \{a_1, \dots, a_{j-1}\} \neq \emptyset \quad \forall j > 1, z \neq 0$;
- (3) $a_i = a_j = q^{2z} (i < j) \Rightarrow \{q^{2(z+1)}, q^{2(z-1)}\} \subset \{a_{i+1}, \dots, a_{j-1}\}$.

Proof. The condition (1) is the identity $y_1 = 1$. Conditions (2),(3) can be proven by induction (see the proof of analogous Theorem 5.1 in [1]). To prove the condition (3) we need the fact that the combinations $(\dots, a_{i-1}, a_i, a_{i+1}, \dots) = (\dots, a, q^{\pm 2}a, a, \dots)$ cannot appear in Λ : the braid relation $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$ is incompatible with the values $\sigma_i = \pm q^{\pm 1}$, $\sigma_{i+1} = \mp q^{\mp 1}$ (see the condition (b) of Proposition 3). \bullet

Consider a Young diagram with $M + 1$ nodes. We place the numbers $1, \dots, M + 1$ into the nodes of the diagram in such a way that these numbers are arranged along rows and columns in ascending order in right and down directions. Such diagram is called a standard Young tableau $[\nu]_{M+1}$. The standard Young tableau $[\nu]_{M+1}$ defines an ascending set of standard tableaux: $[\nu]_1 \subset [\nu]_2 \subset \dots \subset [\nu]_{M+1}$. In addition we associate a number $q^{2(n-m)}$ (the "content") to each node of the standard Young tableau, where (n, m) are coordinates of the node. Example:

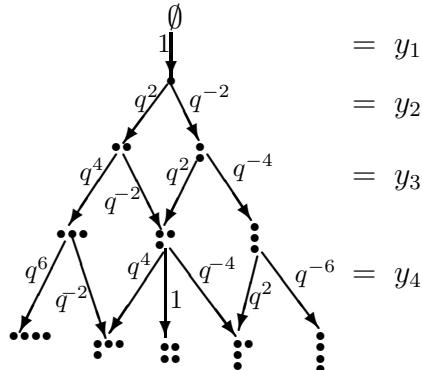
$$\begin{array}{|c|c|c|c|} \hline
 1 & 2 & 4 & 6 \\ \hline
 1 & q^2 & q^4 & q^6 \\ \hline
 3 & 5 & 8 & \\ \hline
 q^{-2} & 1 & q^2 & \\ \hline
 7 & q^{-4} & & \\ \hline
 \end{array}
 \quad \begin{matrix} n \\ \nearrow \\ m \end{matrix} \quad (3.5)$$

In general, for the tableau $[\nu]_{M+1}$, the i -th node $[\nu]_i \setminus [\nu]_{i-1}$ with coordinates (n, m) looks like: $\boxed{i \atop q^{2(n-m)}}$. Thus, to each standard Young tableau $[\nu]_n$ one can associate a string (a_1, \dots, a_n) with $a_i = q^{2(n-m)}$. E.g., a standard Young tableau (3.5) corresponds to a string $(1, q^2, q^{-2}, q^4, 1, q^6, q^{-4}, q^2)$. This string satisfies conditions of Prop. 3 and therefore $(1, q^2, q^{-2}, q^4, 1, q^6, q^{-4}, q^2) \in \text{Spec}(y_1, \dots, y_8)$. This relation between contents of $[\nu]_n$ and elements of $\text{Spec}(y_1, \dots, y_n)$ can be formulated as (cf. Prop. 5.3 [1]):

Proposition 5. *There is a bijection between the set $T(n)$ of the standard Young tableaux with n nodes and the set $\text{Spec}(y_1, \dots, y_n)$.*

4 Coloured Young graph and explicit construction of idempotents e_α

The above results can be visualized in a different form, in terms of a Young graph. By definition, a Young graph is a graph whose vertices are Young diagrams and edges indicate inclusions of diagrams. We put the eigenvalues a_i (colours) of the Jucys-Murphy elements y_i on the edges in such a way that the string (a_1, a_2, \dots, a_n) along the path from the top \emptyset of the Young graph to the diagram λ with n nodes gives the content string of the tableau of shape λ . For example, the coloured Young graph for H_4 is:



The path $\{\emptyset \xrightarrow{1} \bullet \xrightarrow{q^2} \bullet \bullet \xrightarrow{q^{-2}} \bullet \bullet \xrightarrow{1} \bullet \bullet\}$ corresponds to the tableau $[\nu]_4 := \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 1 & q^2 \\ \hline 3 & 4 \\ q^{-2} & 1 \\ \hline \end{array}$ with content string $(1, q^2, q^{-2}, 1)$: the shape of the tableau is given by the shape of the last vertex of the path while the labels of nodes of the tableau shows in which sequence the points \bullet appear in the vertices along the path. The edge indices of the path are eigenvalues of the Jucys-Murphy elements: $(1, q^2, q^{-2}, 1) \in \text{Spec}(y_1, y_2, y_3, y_4)$ corresponding to the values of y_i on the primitive idempotent $e([\nu]_4)$. Thus, we associate a standard Young tableau with n nodes (related to a string in $\text{Spec}(y_1, \dots, y_n)$) and, correspondingly, to the primitive orthogonal idempotent of H_n with a path which starts from the vertex \emptyset and goes down to the vertex with Young diagram with n nodes (the path with n edges in the coloured Young graph). Denote by $X(n)$ the set of all such paths and by $\text{Str}(n)$ the set of the strings $\Lambda = (a_1, \dots, a_n)$ of numbers $a_i = q^{2m_i}$ satisfying conditions (3.4). We collect the above construction in the following statement.

Proposition 6. *There is a bijection between the set $T(n)$ of the standard Young tableaux with n nodes, the set $\text{Spec}(y_1, \dots, y_n)$, the set $\text{Str}(n)$ and the set $X(n)$ of the paths of length n in the Young graph: $T(n) \leftrightarrow \text{Spec}(y_1, \dots, y_n) \leftrightarrow \text{Str}(n) \leftrightarrow X(n)$.*

The dimension of the irreducible representation of $H_n(q)$ (corresponding to the Young diagram λ with n nodes) is equal to the number of standard tableaux $[\nu]_n$ of shape λ or, as we saw, to the number of paths which lead to this Young diagram from the top vertex \emptyset . This number is given by a Frobenius formula $d_\lambda = n!(h_1! \dots h_k!)^{-1} \prod_{i < j} (h_i - h_j)$, where k is the number of rows in λ and h_i are hook lengths of the nodes in the first column of λ (see, e.g., [7]).

Since the coloured Young graph for H_{M+1} contains the whole information about the spectrum of y_k , we can deduce the expressions (in terms of the elements y_k) of all orthogonal primitive idempotents for the Hecke algebra using the inductive procedure proposed in [7]. This special set of primitive orthogonal idempotents has also been described in [4].

Let λ be a Young diagram with $n = n_k$ rows: $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$ and $|\lambda| := \sum_{i=1}^n \lambda_i$ be the number of its nodes. Consider the case when $\lambda_1 = \dots = \lambda_{n_1} = \lambda_{(1)} > \lambda_{n_1+1} = \lambda_{n_1+2} = \dots = \lambda_{n_2} = \lambda_{(2)} > \dots > \lambda_{n_k-n_{k-1}+1} = \dots = \lambda_{|\lambda|} = \lambda_{(n_k)}$:

$$\lambda = \begin{array}{c} \lambda_{(1)} \\ \boxed{n_1} \\ \boxed{n_2-n_1} \\ \vdots \\ n_k-n_{k-1} \end{array} \begin{array}{c} n_1, \lambda_{(1)} \\ n_2, \lambda_{(2)} \\ \dots \\ n_k, \lambda_{(k)} \end{array} \quad (4.1)$$

Here $(n_i, \lambda_{(i)})$ are coordinates of the nodes corresponding to the corners of the diagram λ . Consider any standard Young tableau $[\nu]_{|\lambda|}$ of shape (4.1). Let $e([\nu]_{|\lambda|}) \in H_{|\lambda|}$ be a primitive idempotent corresponding to the tableau $[\nu]_{|\lambda|}$. Taking into account the branching rule implied by the coloured Young graph for $H_{|\lambda|+1}$ we conclude that the following identity holds

$$e([\nu]_{|\lambda|}) \prod_{r=1}^{k+1} (y_{|\lambda|+1} - q^{2(\lambda_{(r)} - n_{r-1})}) = 0,$$

where $\lambda_{(k+1)} = n_0 = 0$. Thus, for a new tableau $[\nu_j]_{|\lambda|+1}$ which is obtained by adding to the tableau $[\nu]_{|\lambda|}$ of shape (4.1) a new node with coordinates $(n_{j-1} + 1, \lambda_{(j)} + 1)$ we obtain the following primitive idempotent (after a normalization)

$$e([\nu_j]_{|\lambda|+1}) := e([\nu]_{|\lambda|}) \prod_{\substack{r=1 \\ r \neq j}}^{k+1} \frac{(y_{|\lambda|+1} - q^{2(\lambda_{(r)} - n_{r-1})})}{(q^{2(\lambda_{(j)} - n_{j-1})} - q^{2(\lambda_{(r)} - n_{r-1})})} = e([\nu]_{|\lambda|}) \Pi_j. \quad (4.2)$$

Using this formula and "initial data" $e(\boxed{1}) = 1$, one can deduce step by step explicit expressions for all primitive orthogonal idempotents for Hecke algebras.

5 q-dimensions for Young diagrams

Consider a linear map $Tr_{d(m+1)}: H_{m+1}(q) \rightarrow H_m(q)$ from the Hecke algebra $H_{m+1}(q)$ to its subalgebra $H_m(q)$ such that $(\forall X, Y \in H_m(q), Z \in H_{m+1}(q))$

$$\begin{aligned} Tr_{d(m+1)}(X) &= z_d X, \quad Tr_{d(m+1)}(X Z Y) = X Tr_{d(m+1)}(Z) Y , \\ Tr_{d(m+1)}(\sigma_m^{\pm 1} X \sigma_m^{\mp 1}) &= Tr_{d(m)}(X) , \quad Tr_{d(m+1)}(\sigma_m) = 1 , \\ Tr_{d(m)} Tr_{d(m+1)}(\sigma_m Z) &= Tr_{d(m)} Tr_{d(m+1)}(Z \sigma_m) , \end{aligned} \quad (5.1)$$

where z_d is a constant which we fix as $z_d = \frac{1-q^{-2d}}{q-q^{-1}}$ for later convenience. Then one can define an Ocneanu's trace $\mathcal{T}r^{(m+1)}: H_{m+1}(q) \rightarrow \mathbf{C}$ as a sequence of maps $\mathcal{T}r^{(m+1)} := Tr_{d(1)} Tr_{d(2)} \cdots Tr_{d(m+1)}$.

Proposition 7. *Ocneanu's traces of idempotents $e([\nu]_{|\lambda|})$, $e([\nu']_{|\lambda|})$ corresponding to tableaux $[\nu]_{|\lambda|}$, $[\nu']_{|\lambda|}$ of the same shape λ coincide. Thus,*

$$\text{qdim}(\lambda) := \mathcal{T}r^{(|\lambda|)} e([\nu]_{|\lambda|}) = \mathcal{T}r^{(|\lambda|)} e([\nu']_{|\lambda|})$$

depends on the diagram λ only.

Using (5.1) we deduce an identity (see Appendix)

$$1 + (q - q^{-1}) \text{Tr}_{d(|\lambda|+1)} \left(\frac{y_{|\lambda|+1} \tau}{1 - y_{|\lambda|+1} \tau} \right) = \frac{(1 - \tau q^{-2d})}{(1 - \tau)} \prod_{k=1}^{|\lambda|} \frac{(1 - \tau y_k)^2}{(1 - q^2 \tau y_k)(1 - q^{-2} \tau y_k)}, \quad (5.2)$$

where τ is a parameter. To calculate "qdim" for the diagram (4.1) we need to find the value of the element (5.2) on the idempotent $e([\nu]_{|\lambda|})$, where $[\nu]_{|\lambda|}$ is any Young tableau of shape (4.1). We take the "row-standard" tableau $[\nu]_{|\lambda|}$ corresponding to the eigenvalues of y_k arranged along the rows from left to right and from top to bottom:

$$\begin{aligned} y_1 &= 1, \quad y_2 = q^2, \quad y_3 = q^4, \quad \dots, \quad y_{\lambda_1-1} = q^{2(\lambda_1-2)}, \quad y_{\lambda_1} = q^{2(\lambda_1-1)}, \\ y_{\lambda_1+1} &= q^{-2}, \quad y_{\lambda_1+1} = 1, \quad \dots, \quad y_{\lambda_1+\lambda_2} = q^{2(\lambda_2-2)}, \\ &\dots, \\ y_{|\lambda|-\lambda_n+1} &= q^{-2(n-1)}, \quad \dots, \quad y_{|\lambda|} = q^{2(\lambda_n-n)}. \end{aligned}$$

The result is ($n_k = n, n_0 := 0$)

$$\text{Tr}_{d(|\lambda|+1)} \left(\sum_j P_j \frac{(q - q^{-1}) \mu_j \tau}{1 - \mu_j \tau} \right) = e([\nu]_{|\lambda|}) \left(\frac{1 - \tau q^{-2d}}{1 - \tau q^{-2n}} \prod_{j=1}^k \frac{1 - \tau q^{2(\lambda_j-n_j)}}{1 - \tau \mu_j} - 1 \right), \quad (5.3)$$

where we have inserted into the l.h.s. the spectral decomposition of the idempotent $e([\nu]_{|\lambda|})$ (see (4.2)):

$$e([\nu]_{|\lambda|}) = e([\nu]_{|\lambda|}) \sum_j \Pi_j = \sum_j P_j, \quad P_j y_{|\lambda|+1} = P_j q^{2(\lambda_j-n_{j-1})} = P_j \mu_j.$$

The operator P_j projects $y_{|\lambda|+1}$ on its eigenvalue $\mu_j := q^{2(\lambda_j-n_{j-1})}$ which appeared in the denominator of the r.h.s. of (5.3). Comparing both sides of eq. (5.3) we deduce

$$\begin{aligned} \text{Tr}_{d(|\lambda|+1)} (P_j) &= e([\nu]_{|\lambda|}) \lim_{\tau \rightarrow 1/\mu_j} \frac{(1 - \mu_j \tau)}{(q - q^{-1})} \left(\frac{1 - \tau q^{-2d}}{1 - \tau q^{-2n}} \prod_{r=1}^k \frac{1 - \tau q^{2(\lambda_r-n_r)}}{1 - \tau \mu_r} \right) \\ &= e([\nu]_{|\lambda|}) \cdot q^{-d} [q^{(\lambda_j-n_{j-1}+d)}]_q \frac{\prod_{n,m \in \lambda} [h_{n,m}]_q}{\prod_{n,m \in \lambda^{(j)}} [h_{n,m}]_q}, \end{aligned} \quad (5.4)$$

where $h_{n,m}$ are hook lengths of nodes (n, m) of the diagrams λ or $\lambda^{(j)}$ ($\lambda^{(j)}$ is a diagram obtained by adding to the diagram λ a new node with coordinates $(n_{j-1} + 1, \lambda_{(j)} + 1)$). Applying the Ocneanu's trace $\mathcal{T}r^{(|\lambda|)}$ to eq. (5.4) we find a recurrent relation:

$$\text{qdim}(\lambda^{(j)}) = \text{qdim}(\lambda) q^{-d} [\lambda_{(j)} - n_{j-1} + d]_q \frac{\prod_{n,m \in \lambda} [h_{n,m}]_q}{\prod_{n,m \in \lambda_j} [h_{n,m}]_q},$$

which is solved by

$$\text{qdim}(\lambda) = q^{-d|\lambda|} \prod_{n,m \in \lambda} \frac{[d+m-n]_q}{[h_{n,m}]_q}.$$

Up to a normalization factor this formula has firstly been obtained in [5].

For R -matrix representations of $H_{M+1}(q)$ (about R -matrix representations of the Hecke algebra see Refs. [9], [10]) which corresponds to the quantum supergroup $GL_q(N|M)$, the parameter d equals $N - M$. This justifies our choice of the parametrization of z_d in the first eq. of (5.1).

Proposition 7 can be generalized. Let T be a quantum matrix satisfying

$$\hat{R}_{12} T_1 T_2 = \hat{R}_{12} T_1 T_2 \quad (5.5)$$

in the notations of [10], where $\hat{R}_{12} = \rho(\sigma_1)$ is the R -matrix representation of the Hecke algebra.

Proposition 8. *The quantum traces (for the definition of the quantum trace see e.g. [10], [11], [12]) of the matrices $[T_1 \cdots T_{|\lambda|} \rho(e([\nu]_{|\lambda|}))]$ and $[T_1 \cdots T_{|\lambda|} \rho(e([\nu']_{|\lambda|}))]$*

$$\chi_\lambda(T) := Tr_{R(1 \rightarrow |\lambda|)} (T_1 \cdots T_{|\lambda|} \rho(e([\nu]_{|\lambda|}))) = Tr_{R(1 \rightarrow |\lambda|)} (T_1 \cdots T_{|\lambda|} \rho(e([\nu']_{|\lambda|}))),$$

corresponding to tableaux $[\nu]_{|\lambda|}$ and $[\nu']_{|\lambda|}$ of the same shape λ , coincide. Thus, $\chi_\lambda(T)$ depends only on the diagram λ .

Consider the $GL_q(N)$ quantum group (5.5) with a standard $GL_q(N)$ Drinfeld-Jimbo R -matrix \hat{R}_{12} [10]. It is known [9], [10] that the standard $GL_q(N)$ matrix \hat{R}_{12} defines the representation of the Hecke algebra. We note that the $GL_q(N)$ quantum matrix T can be realized by arbitrary numerical diagonal ($N \times N$) matrix X . Then $\chi_\lambda(X)$ is a numerical function of the deformation parameter q and the entries of X . In the classical limit $q \rightarrow 1$ the operator $\rho(e([\nu]_{|\lambda|}))$ tends to the Young projector and the function $\chi_\lambda(X)$ coincides with a character of the element X ($X \in GL(N)$) in the representation corresponding to the diagram λ .

6 Appendix

Taking into account the definition of the generators y_m we have equations

$$\frac{1}{(t - y_{m+1})} \sigma_m^{-1} = \sigma_m^{-1} \frac{1}{(t - y_m)} + \frac{(q - q^{-1})y_m}{(t - y_{m+1})} \frac{1}{(t - y_m)} \quad (6.1)$$

$$\frac{1}{(t - y_{m+1})} \sigma_m = \sigma_m^{-1} \frac{1}{(t - y_m)} + \frac{(q - q^{-1})t}{(t - y_m)} \frac{1}{(t - y_{m+1})}. \quad (6.2)$$

Eqs. (6.1), (6.2) and the definition of the map (5.1) give a recurrent relation

$$\frac{(t - q^2 y_m)(t - q^{-2} y_m)}{(t - y_m)^2} Z_{m+1} = Z_m + \frac{(q - q^{-1})y_m}{(t - y_m)^2} [1 - (q - q^{-1})z_d], \quad (6.3)$$

where the parameter z_d is introduced in (5.1) and

$$Z_m := \text{Tr}_{d(m)} \left(\frac{1}{(t - y_m)} \right).$$

Eq. (6.3) is simplified by the substitution $Z_m = \tilde{Z}_m - [1 - (q - q^{-1})z_d] / ((q - q^{-1})t)$ and we have

$$\frac{(t - q^2 y_m)(t - q^{-2} y_m)}{(t - y_m)^2} \tilde{Z}_{m+1} = \tilde{Z}_m.$$

This equation can be easily solved and finally we obtain the expression

$$\begin{aligned} Z_{m+1} &= \frac{1}{(q - q^{-1})t} \left(1 + \frac{(q - q^{-1}) z_d}{(t - 1)} \right) \prod_{k=1}^m \frac{(t - y_k)^2}{(t - q^2 y_k)(t - q^{-2} y_k)} \\ &\quad - \frac{1}{(q - q^{-1})t} [1 - (q - q^{-1})z_d], \end{aligned}$$

which is equivalent to (5.2) for $t = 1/\tau$.

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